Module-3: Nonlinear First-Order Partial Differential Equations

1. Introduction

In this module, we discuss nonlinear partial differential equations of first-order of the form

$$F(x,y,z,p,q) = 0 \tag{1}$$

where x, y are independent variables, $z = z(x,y)$ and $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$, the function F being nonlinear. We have already seen in previous study that the partial differential equation arising from the two-parameter system of surfaces

$$f(x,y,z,a,b) = 0 \tag{2}$$

is of this form. We shall see later that the converse is also true.

Let us define first different types of solutions involved for the equation (1) and consider a geometrical approach, introduced by Cauchy, to find the solutions.

2. Definitions of Various Types of Solutions

Any envelope of the system (2) touches at each of its points a member of the system and, therefore, it has the same set of values (x,y,z,p,q) as the particular surface. Thus, it is also a solution of the partial differential equation (1). Hence, we are led to the following three types of integrals of the equation (1).

(a) **Complete integral**

The two-parameter system of surfaces of the form (2) is called a *complete integral* of the equation (1).

(b) **General integral**

Suppose there exists a relation between the parameters a and b of the form $b = \psi(a)$, ψ being arbitrary. Then the one-parameter subsystem $f(x,y,z,a,\psi(a)) = 0$ of the system (2) forms its envelope and is called the *general integral* of (1).

(c) **Singular integral**

If the envelope of the two-parameter system of surfaces (2) exists, then it is also a solution of the equation (1) and is known as the *singular integral*.

Example 1: Verify that $z = ax + by + a + b - ab$ is a complete integral of the partial differential equation $z = px + qy + p + q - pq$, a and b being arbitrary constants.

Show also that the envelope of all planes corresponding to complete integrals provided singular integral of the differential equation and determine a general integral by finding the envelope of these planes that pass through the origin.

Solution: Let $f(x,y,z,a,b) = z-(ax+by+a+b-ab) = 0$ so that $p = \frac{\partial z}{\partial x} = a$, $q = \frac{\partial z}{\partial y} = b$ and hence $f(x,y,z,a,b) = 0$ is a complete integral of the equation $z = px + qy + p + q - pq$.

Now, the envelope of the two-parameter system $f(x,y,z,a,b) = 0$ is obtained by eliminating a and b from the equations $f = 0$, $\frac{\partial f}{\partial a} = 0$, $\frac{\partial f}{\partial b} = 0$, i.e. from

$$z = ax + by + a + b - ab, \quad -(x+1) + b = 0, \quad -(y+1) + a = 0.$$

Elimination of a and b from these equations gives $z = (x+1)(y+1)$ which is the singular integral.

To find the general integral, we suppose that there exists a relation of the form $b = \psi(a)$ between the parameters a and b. So, we consider the one-parameter system

$$f(x,y,z,a,\psi(a)) = z - \{ax + \psi(a)y + a + \psi(a) - a\psi(a)\} = 0.$$

$$\text{Thus} \quad \frac{\partial f}{\partial a} = -\{x + \psi'(a)y + 1 + \psi'(a) - \psi(a) - a\psi'(a)\} = 0$$

Since the envelope passes through the origin, so from equation $f = 0$, we get

$$-a - \psi(a) + a\psi(a) = 0 \Rightarrow \psi(a) = \frac{a}{a-1} \Rightarrow \psi'(a) = -\frac{1}{(a-1)^2}$$

Then the equation $\frac{\partial f}{\partial a} = 0$ gives $x - \frac{1}{(a-1)^2}y + 1 - \frac{1}{(a-1)^2} - \frac{a}{a-1} + \frac{a}{(a-1)^2} = 0 \Rightarrow a = \sqrt{\frac{y}{x}} + 1$ so that $\psi(a) = \frac{\sqrt{y}+\sqrt{x}}{\sqrt{y}}$.

Thus, from the equation $f = 0$, we get by substituting the values of a and $\psi(a)$

$$z = \frac{\sqrt{y}+\sqrt{x}}{\sqrt{y}} \cdot x + \frac{\sqrt{y}+\sqrt{x}}{\sqrt{y}} \cdot y + \frac{\sqrt{y}+\sqrt{x}}{\sqrt{y}} - \frac{\sqrt{y}}{\sqrt{x}} \cdot \frac{\sqrt{y}+\sqrt{x}}{\sqrt{y}}$$

$$\Rightarrow \quad z = 2\sqrt{xy} + x + y, \ \text{ i.e. } (x+y-z)^2 = 4xy.$$

This is the required general integral.

3. Cauchy's Method of Characteristics

We now introduce a method, due to Cauchy, based on geometrical idea, to solve nonlinear partial differential of the form (1).

Suppose the plane through the point $P(x_0, y_0, z_0)$ with its normal parallel to the direction $\hat{\mathbf{n}}$ with direction ratios $(p_0, q_0, -1)$ be uniquely specified by the set of numbers $(x_0, y_0, z_0, p_0, q_0)$. Conversely, any set of five real numbers defines a plane in three-dimensional space. A plane element $(x_0, y_0, z_0, p_0, q_0)$ satisfying the equation (1) is called an *integral element*.

We rewrite equation (1) in the form

$$q = G(x, y, z, p) \tag{3}$$

and keep x, y, z fixed but vary p so that we obtain a set of plane elements $\{x_0, y_0, z_0, p, G(x_0, y_0, z_0, p)\}$ depending on the single parameter p only and passes through the point P. Thus the planar elements envelope a cone with P as vertex, called the *elementary cone* of the equation (1) at the point P.

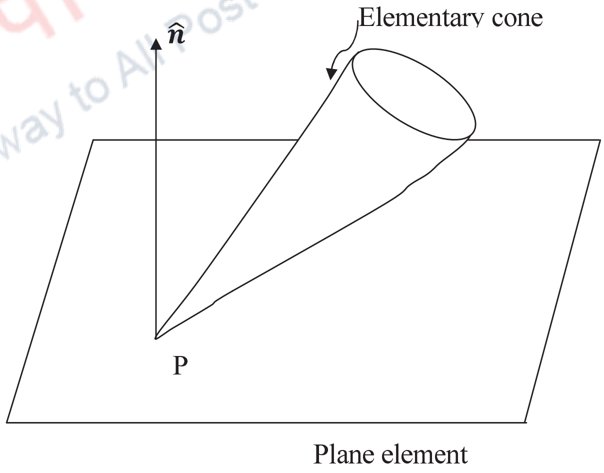

Fig. 1

Now consider a surface S given by the equation

$$z = g(x, y) \tag{4}$$

where the function $g(x, y)$ and its first partial derivatives with respect to x and y are assumed to be continuous in a region Ω of the xy-plane. Then the tangent plane at every point of S determines a plane element of the type $\left\{x_0, y_0, g(x_0, y_0), g_x(x_0, y_0), g_y(x_0, y_0)\right\}$

which is called the *tangent element* of the surface S at the point $(x_0, y_0, g(x_0, y_0))$. Thus, we have the result:

Theorem 1: *A necessary and sufficient condition for a surface to be an integral surface of a partial differential equation is that at each point its tangent element should touch the elementary cone.*

Cauchy characteristic equations

Consider a curve Γ with parametric equations $x = x(t)$, $y = y(t)$, $z = z(t)$ lying on the surface S given by equation (4) so that

$$z = g\{x(t), y(t)\}, \ \ \forall\, t \in I, \tag{5}$$

where I is the given interval. Then, if P_0 is a point on Γ determined by the parameter t_0. the direction ratios of the tangent line $\overline{P_0P_1}$ are $\{x'(t_0), y'(t_0), z'(t_0)\}$, where $x'(t_0) = \left(\frac{dx}{dt}\right)_{t=t_0}$ etc. This direction is perpendicular to the direction $(p_0, q_0, -1)$ provided

$$p_0x'(t_0) + q_0y'(t_0) + (-1)z'(t_0) = 0, \ \text{i.e} \ \ z'(t_0) = p_0x'(t_0) + q_0y'(t_0).$$

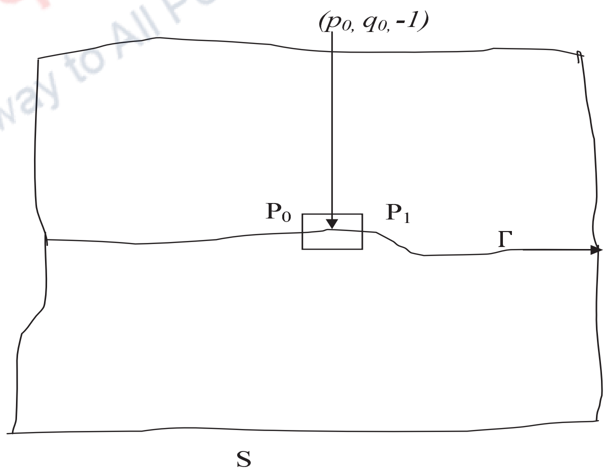

Fig. 2

Thus, any set $\{x(t), y(t), z(t), p(t), q(t)\}$ of five real functions satisfying the condition

$$z'(t) = p(t)x'(t) + q(t)y'(t) \tag{6}$$

defines a strip at the point (x, y, z) of the curve Γ. If such a strip is an integral element of the equation (1), viz. $F(x, y, z, p, q) = 0$, then it is called an *integral strip* of the equation. Thus the set of functions $\{x(t), y(t), z(t), p(t), q(t)\}$ is an integral strip of this partial differential equation (1), if they satisfy the condition (6) and the condition

$$F\{x(t), y(t), z(t), p(t), q(t)\} = 0 \ \ \forall \ t \in I. \tag{7}$$

If at each point the curve Γ touches a generator of the elementary cone, then the corresponding strip is called a *characteristic strip*. The point $(x + dx, y + dy, z + dz)$ lies on the tangent plane to the elementary cone if

$$dz = pdx + qdy \tag{8}$$

where p, q satisfy the equation (1). Now differentiating (1) and (8) with respect to p, we get respectively

$$\frac{\partial F}{\partial p} + \frac{\partial F}{\partial q}\frac{\partial q}{\partial p} = 0 \ \ \text{and} 0 = dx + \frac{dq}{dp}dy. \tag{9}$$

Using relations (8) and (9), we have

$$\frac{dx}{F_p} = \frac{dy}{F_q} = \frac{dz}{pF_p + qF_q}. \tag{10}$$

Thus, $x'(t)$, $y'(t)$, $z'(t)$ are proportional to F_p, F_q, $pF_p + qF_q$ respectively along a characteristic strip. We choose the parameter t such that

$$x'(t) = F_p, \ \ y'(t) = F_q, \ \ z'(t) = pF_p + qF_q.$$

Now along a characteristic strip $p = p(t)$, a function of t and, therefore,

$$p'(t) = \frac{\partial p}{\partial x}x'(t) + \frac{\partial p}{\partial y}y'(t) = \frac{\partial p}{\partial x}\frac{\partial F}{\partial p} + \frac{\partial p}{\partial y}\frac{\partial F}{\partial q} = \frac{\partial p}{\partial x}\frac{\partial F}{\partial p} + \frac{\partial q}{\partial x}\frac{\partial F}{\partial q},$$

where we used the result $\frac{\partial p}{\partial y} = \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial q}{\partial x}$. Also, differentiation of (7) partially with respect to x gives

$$\frac{\partial F}{\partial x} + p\frac{\partial F}{\partial z} + \frac{\partial F}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial F}{\partial q}\frac{\partial q}{\partial x} = 0$$

$$\text{i.e.} \qquad \frac{\partial F}{\partial x} + p\frac{\partial F}{\partial z} + p'(t) = 0$$

so that on a characteristics strip

$$p'(t) = -(F_x + pF_z).$$

Similarly, we have $q'(t) = -(F_y + qF_z)$.

Thus, we have the following system of five ordinary differential equations for the determination of the characteristic strip:

$$\begin{aligned} &x'(t) = F_p, \;\; y'(t) = F_q, \;\; z'(t) = pF_p + qF_q, \\ &p'(t) = -(F_x + pF_z), \;\; q'(t) = -(F_y + qF_z). \end{aligned} \tag{11}$$

Equations (11) are known as *Cauchy's characteristic equations* of the partial differential equation $F(x,y,z,p,q) = 0$.

Theorem 2: *The function $F(x,y,z,p,q)$ is constant along every characteristic strip of the partial differential equation $F(x,y,z,p,q) = 0$.*

Proof: Along a characteristic strip

$$\begin{aligned} &\frac{d}{dt}\Big[F\Big\{x(t), y(t), z(t), p(t), q(t)\Big\}\Big] \\ &= F_x x'(t) + F_y y'(t) + F_z z'(t) + F_p p'(t) + F_q q'(t) \\ &= F_x F_p + F_y F_q + F_z(pF_p + qF_q) - F_p(F_x + pF_z) - F_q(F_y + qF_z) \\ &= 0 \end{aligned}$$

so that $F(x,y,z,p,q)$ is constant along characteristic strip.

Corollary: If a characteristic strip contains at least one integral element of $F(x,y,z,p,q) = 0$, then it is an integral strip of this equation.

We are now in a position *to solve Cauchy problem* stated earlier.
Let us find the solution of the equation (1), viz. $F(x,y,z,p,q) = 0$ such that the integral surface passes through the curve Γ having parametric equations

$$x = \phi(\xi), \;\; y = \psi(\xi), \;\; z = \chi(\xi). \tag{12}$$

Then, in the solution

$$x = x(x_0, y_0, z_0, p_0, q_0, t_0, t) \text{ etc} \tag{13}$$

of the characteristic equations (11), we may take the initial values of x, y, z as $x_0 = \phi(\xi)$, $y_0 = \psi(\xi)$, $z_0 = \chi(\xi)$ and so the corresponding values of p_0, q_0 are given by the relations

$$\chi'(\xi) = p_0\phi'(\xi) + q_0\psi'(\xi) \;\text{ and }\; F\{\phi(\xi), \psi(\xi), \chi(\xi), p_0, q_0\} = 0$$

Substituting these values of x_0, y_0, z_0, p_0, q_0 and the appropriate value of t_0 in equation (13), x, y and z can be expressed in terms of two parameters ξ and t in the form

$$x = X(\xi, t), \quad y = Y(\xi, t) \quad z = Z(\xi, t).$$

Elimination of ξ and t amongst these relations leads to an equation of the form

$$\Theta(x, y, z) = 0 \tag{14}$$

This is the equation of the integral surface of the equation $F(x, y, z, p, q) = 0$ through the curve Γ.

Example 2: Find the characteristics of the equation $pq = z$ and determine the integral surface which passes through the parabola $x = 0, y^2 = z$.

Solution: Here $F(x, y, z, p, q) = pq - z = 0$ and so the characteristic equations are

$$x'(t) = F_p = q, \quad y'(t) = F_q = p, \quad z'(t) = pF_p + qF_q = 2pq,$$
$$p'(t) = -(F_x + pF_z) = p, \quad q'(t) = -(F_y + qF_z) = q$$

Since the equation of the given curve is $x = 0$, $y^2 = z$, so we choose the initial values as $x_0 = 0$, $y_0 = \xi$ $z_0 = \xi^2$.

From the equation $z_0' = p_0x_0' + q_0y_0'$, we get $q_0 = 2\xi$ and the given equation $pq = z$ provides $p_0 = \xi/2$.

Now, the equations $x'(t) = q$ and $q'(t) = q$ on integration, give $x = q + c_1$, while the equations $y'(t) = p$ and $p'(t) = p$, on integration, lead to $y = p + c_2$, where c_1 and c_2 are constants. Using the initial conditions, we get $0 = 2\xi + c_1$ and $\xi = \xi/2 + c_2$ so that $c_1 = -2\xi$ and $c_2 = \xi/2$. Hence, it follows that

$$x = q - 2\xi, \quad y = p + \xi/2.$$

Again, the equations $p'(t) = p$ and $q'(t) = q$, on integration, give $p = c_3e^t$ and $q = c_4e^t$, where c_3 and c_4 are constants. Using the initial conditions, we have $\xi/2 = c_3$, $2\xi = c_4$. Thus $p = \frac{\xi}{2}e^t$, $q = 2\xi e^t$. Hence,

$$x = 2\xi(e^t - 1), \quad y = \frac{\xi}{2}(e^t + 1) \Rightarrow e^t = \frac{4y + x}{4y - x}, \quad \xi = \frac{1}{4}(4y - x).$$

Also, the characteristic equation $z'(t) = 2pq = 2\xi^2e^{2t}$ gives on integration $z = \xi^2e^{2t}$, where we have used the initial condition $z_0 = \xi^2$ at $t = 0$. Thus

$$z = \frac{(4y - x)^2}{16} \cdot \frac{(4y + x)^2}{(4y - x)^2}, \quad \text{i.e} \quad 16z = (x + 4y)^2$$

Hence the characteristics of the given partial differential equation are

$$x = 2\xi(e^t - 1), \quad y = \frac{\xi}{2}(e^t + 1), \quad z = \xi^2 e^{2t}$$

and the equation of the required integral surface is $16z = (x + 4y)^2$